LAB REPORT: LAB 5

TNM079, MODELING AND ANIMATION

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Abstract

This is the fifth lab in a series of six labs in the course TNM079 Modeling and Animation at Linköping University. The report covers the topic of level-set methods in computer graphics and the implementation of a level-set framework. A level-set is a subset of an implicit surface, which by solving a number of partial differential equations (PDEs) can be deformed. The implicit representation enables easy changes in topology and formation. Stable schemes on both hyperbolic and parabolic differentials are discussed. They allow for erosion, dilation and advection.

1 Introduction

Level sets are a subset of an implicit function. They are often used in conjunction with deformation operations, such as erosion, dilation or advection. The method represents an interface as a level set S of the level set function ϕ as

$$S = \{x \in \Re^d : \phi(\mathbf{x}) = h\} \tag{1}$$

where points are inside S when $\phi(\mathbf{x}) < h$, and outside when $\phi(\mathbf{x}) > h$.

2 Background

It can be shown that the normal and curvature of any level set of ϕ is

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} \tag{2}$$

$$\kappa = \nabla \cdot \mathbf{n} \tag{3}$$

To enable deformation of this implicit geometry, equations of motion are derived for the level set function by introducing time dependence to Equation 1. The first way to do this is to vary the iso-value h over time, such that $S(t) = \{x \in \Re^d : \phi(\mathbf{x}) = h(t)\}$. This is called *the static level set formulation*.

The evolution of level sets is described of a function as the iso-value changes. The level sets can however not intersect by definition, which imposes a great constraint on the deformations possible. The second way to introduce time dependence is to vary the level set function itself over time, such that $S(t) = \{x \in \Re^d : \phi(\mathbf{x}, t) = h\}$. A point $\alpha(t)$ is selected and studied in order to derive equations of motion for S. Since $\alpha(t)$ is on S, it can be concluded that $\phi(\alpha(t), t) = h$. If this is differentiated with respect to time, Equation 4 is obtained.

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= -\nabla\phi \cdot \frac{d\alpha}{dt} \\ &= -F|\nabla\phi|\end{aligned} \tag{4}$$

F is referred to as the *level set speed function*:

$$F = \mathbf{n} \cdot \frac{d\alpha}{dt} = \frac{\nabla\phi}{|\nabla\phi|} \cdot \frac{d\alpha}{dt} \tag{5}$$

This can be interpreted as the speed $d\alpha/dt$ at a point α, projected onto the normal at that point. In other words, the speed in the normal direction. Equation 4 provides the user with the means of modifying and manipulating the level set function, to achieve a desired motion of the surface S.

Using $h = 0$ has some benefits, as it allows the definition of *inside* and *outside* the surface to be carried out through sign convention. Equation 4 is a continuous derivation, but in order to apply the theory in computer graphics both the temporal and spatial domain have to be discretized. This could possibly have a number of numerical implications, depending on what PDE is used. The temporal discretization determines how Equation 4 evolves in time. Since the time needs to be discrete, the PDE is evolved using discrete time steps of Δt, using either an implicit or explicit integration scheme. Regardless of the time step Δt, implicit schemes are unconditionally stable. However, since the computation of these schemes relies on solving a major number of equations, they are difficult and computationally demanding. Explicit methods are simple to implement, but does however suffer from stability constraints imposed on the time step. Explicit methods are often used for solving level set equations, despite this constraint. An explicit scheme is the forward Euler given by

$$\frac{\partial \phi}{\partial t} \approx \frac{\phi^{n+1} - \phi^n}{\Delta t} \tag{6}$$

where ϕ^n defines the values of ϕ at time instance t^n, and ϕ^{n+1} at the time instance $t^n + \Delta t$. For better accuracy, the Euler scheme could be changed to a *total variation diminishing Runge-Kutta method*, but however at the cost of more computations.

The spatial discretization relies on what PDE is used and thus will two fundamental types be used, *hyperbolic* and *parabolic*. For hyperbolic advection two versions of Equation 4 are used

$$\frac{\partial \phi}{\partial t} = -\mathbf{V} \cdot \nabla \phi = -F|\nabla \phi| \tag{7}$$

Equation 7 means advecting the interface in a vector field V or in the direction of the surface normal. This equation can be used for erosion and dilation, in other words describing a flow of information in a specific direction. Only the sample points behind, or *up-wind* to, the current position should be used in the discretization, which results in a *finite difference (FD) approximation*

$$\frac{\partial \phi}{\partial x} \approx \begin{cases} \phi_x^+ = (\phi_{i+1,j,k} - \phi_{i,j,k})/\Delta x & \text{if } V_x < 0 \\ \phi_x^+ = (\phi_{i,j,k} - \phi_{i-1,j,k})/\Delta x & \text{if } V_x > 0 \end{cases} \tag{8}$$

In Equation 8, the direction of the flow is not known. To deduce this, Godunov's method can be used to evaluate partial derivatives. We get a *first order accurate* approximation

$$\left(\frac{\partial \phi}{\partial x}\right)^2 \approx \begin{cases} max[max(\phi_x^-, 0)^2, min(\phi_x^+, 0)^2] F > 0 \\ max[min(\phi_x^-, 0)^2, max(\phi_x^+, 0)^2] F < 0 \end{cases} \tag{9}$$

The parabolic type is often used to smooth deformations and can be expressed as

$$\frac{\partial \phi}{\partial t} = \alpha \kappa |\nabla \phi| \tag{10}$$

in which α is a scaling parameter and κ is curvature. This equation compared to the hyperbolic representation is that it has no certain direction. Therefore, a second order accurate central difference scheme for space is needed

$$\frac{\partial \phi}{\partial x} \approx \phi_x^{\pm} = \frac{\phi_{i+1,j,k} - \phi_{i-1,j,k}}{2\Delta x} \tag{11}$$

And a second order central difference scheme as can be seen in the two following Equations 12 and 13.

$$\frac{\partial^2 \phi}{\partial^2 x} \approx \frac{\phi_{i+1,j,k} - 2\phi_{i,j,k} + \phi_{i-1,j,k}}{\Delta x^2} \tag{12}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} \approx \frac{\phi_{i+1,j+1,k} - \phi_{i+1,j-1,k} + \phi_{i-1,j-1,k} - \phi_{i-1,j+1,k}}{4\Delta x \Delta y} \tag{13}$$

The stability constraints required in the hyperbolic type for the time step Δt is not necessary in this case as information travels at infinite speed. ϕ needs to be continuous with defined gradients in order for use of Equation 4. However, because of the spatial discretization, this requirement can be relaxed. The gradients of ϕ can be discontinuous, although the *rate-of-change* of ϕ must be bounded by the finite Lipschitz constant C. To assure stability, $C = 1$, meaning that ϕ has to satisfy the *Eikonal equation*

$$|\nabla \phi| = 1 \tag{14}$$

3 Results

The first task was to implement the differential equations with both a forward, backward and central difference scheme using the equations given in 8 and 10. The second derivative scheme was also implemented using Equation 11. These equations were later used when evaluating the advection of a vector field on the mesh. The backward-scheme is used to calculate the gradient's first value when the vector field travels backwards on the x axis. By individually evaluating each variable of the vector field, a differing scheme can be selected. If a deformation of the mesh along the surface normal is desired, Equation 7 will be used. To evaluate the sign of the second derivative the Godunov scheme is used. Figure 1-4 performs a closing, which is when a dilation is performed followed by an erosion. By doing this, the sub-meshes of the model can be removed. In these figures one can notice how the surrounding dots are removed after the closing.

4 Lab partner and grade

My lab partner was Viktor Tholén, and I am aiming for grade 3.

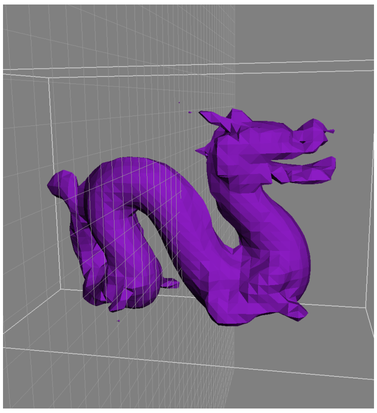

Figure 1: A broken model with sub-meshes surrounding it.

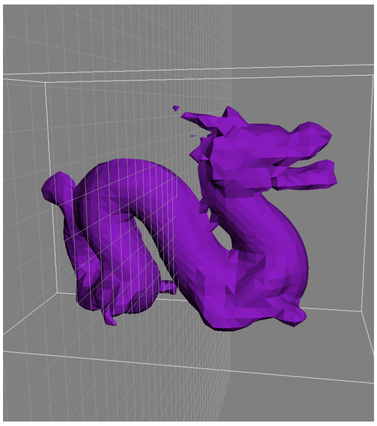

Figure 2: The model after dilation.

References

[1] Emma Broman Mark Eric Dieckmann, Robin Skånberg. *Level-sets*. Modeling and animation, lab 5, 2021.

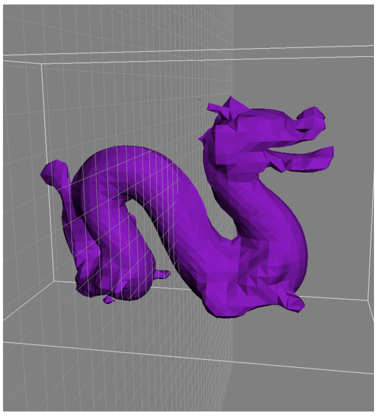

Figure 3: The model after erosion.

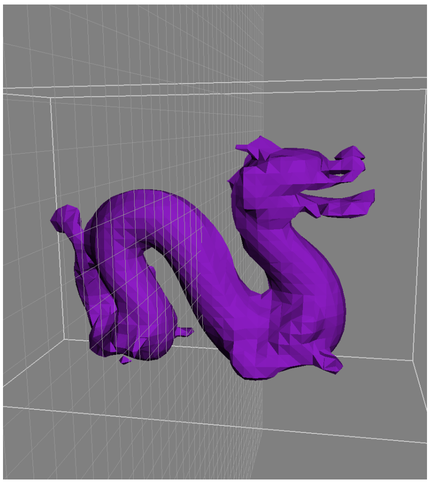

Figure 4: The model after closing to smooth out the surface.