LAB REPORT: LAB 4

TNM079, MODELING AND ANIMATION

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Monday 13th September, 2021

Abstract

This is the fourth lab in a series of six labs in the course TNM079 Modeling and Animation at Linköping University. The report covers the topic of implicit surfaces in computer graphics and the implementation of implicit surfaces through a modeling framework called constructive solid geometry (CSG). Operations such as union, difference or intersection are used. In this lab, only analytical surfaces such as planes, cuboid or spheres will be processed.

1 Introduction

In computer graphics, implicit surface representation can be seen as an indirect way of representing surfaces. In explicit surface representations, like for example meshes, the elements of the surface directly define the surface. However, for the implicit or indirect surface representation the surface is defined by an equation. This equation needs to be solved in order to find the surface geometry.

2 Background

In this lab and report, to show the difference between explicit and implicit representations, the analytical circle with radius R is used. The explicit representation of this circle is given by Equation 1.

$$x = \pm\sqrt{R^2 - y^2} \tag{1}$$

This equation will give a value for x for every value of $y[R, R]$, such that the points $[x, y]$ and $[x, y]$ are positioned on the circle. The implicit circle is described by the Equation 2.

$$f(x, y) = x^2 + y^2 \tag{2}$$

The function assigns a scalar to every point in the x-y plane. A specific iso-value, C, in this scalar field needs to be studied in order to find the circle. A subset of the entire x-y domain can be cut out by choosing all points in the plane where $f(x, y) = C$. This subset is called a level-set of the implicit function $f(x, y, R)$.

The iso-value $C = R^2$ finds the level-set that describes the circle in Equation 2. The set of points $[x, y]$ that satisfies this equation is the implicit representation of a circle with the radius R. The level set of a scalar function will have the dimension of the scalar function minus 1. A 2D scalar function turns into a 1D contour line.

With implicit representations, it is easy to define points that are either inside, outside or on the surface. Given an iso-value $C = f(x)$, a point x can be classified as follows

$$\begin{aligned} Inside &: f(\mathbf{x}) < C \\ Outside &: f(\mathbf{x}) > C \\ Onsurface &: f(\mathbf{x}) = C \end{aligned} \tag{3}$$

This property is used in the intersection or union operations. Differential attributes can efficiently be computed with implicit surfaces, for example surface normals. The expression of the normal can be derived of a 3D scalar field by studying the gradient ∇ of the scalar field $f(x,y,z)$. Assuming that a coordinate frame can be found where one of the basis vectors points in the normal direction Equation 4 can be obtained.

$$\nabla = \vec{e_1}\frac{\partial}{\partial\vec{e_1}} + \vec{e_2}\frac{\partial}{\partial\vec{e_2}} + \vec{n}\frac{\partial}{\partial\vec{n}} = \vec{e_1}(\vec{e_1}\cdot\nabla) + \vec{e_2}(\vec{e_2}\cdot\nabla) + \vec{n}(\vec{n}\cdot\nabla) \tag{4}$$

This gives

$$\nabla f = \vec{e_1}(\vec{e_1}\cdot\nabla f) + \vec{e_2}(\vec{e_2}\cdot\nabla f) + \vec{n}(\vec{n}\cdot\nabla f) \tag{5}$$

If the level-set is a manifold, it can be assumed that an infinitesimally small patch of the surface will be flat. Since $\vec{n}$ is defined to be in the direction of the surface normal and also the manifold is locally flat it can be assumed that $\vec{e_1}$ and $\vec{e_2}$ both will be located in the tangent plane of the surface and on the surface itself. Therefore, the scalar field f will be locally constant everywhere in the tangent plane and Equation 6 is given.

$$\begin{aligned} \frac{\partial f}{\partial\vec{e_1}} &= \frac{\partial f}{\partial\vec{e_2}} = 0 \\ \rightarrow \nabla f &= \vec{n}(\vec{n}\cdot\nabla f) \\ \rightarrow \vec{n} &= \pm\frac{\nabla f}{|\nabla f|} \\ \rightarrow \vec{n} &= \frac{\nabla f}{|\nabla f|} \end{aligned} \tag{6}$$

when using the sign convention in Equation 4. It can now be noted that as long as the iso-surfaces of a scalar field f are manifolds, the normalized gradient of a scalar field is the surface normal to the manifold that intersects that point.

A discrete scheme is required to calculate the differential of an implicit surface. Two types of schemes are forward-differential (Equation 8) or central difference approximation (Equation **??**).

$$D_x = \lim_{h\to\infty}\frac{f(x_0+h)-f(x_0)}{h} \approx \frac{f(x_0+\epsilon)-f(x_0)}{\epsilon} \tag{7}$$

$$D_x = \lim_{h\to\infty}\frac{f(x_0+h)-f(x_0)}{h} \approx \frac{f(x_0+\epsilon)-f(x_0-\epsilon)}{2\epsilon} \tag{8}$$

Gradients can now be calculated with the discrete differential operator, which enables the evaluation of both normals and curvature at any point in the scalar field described by $f(x,y,z)$.

The second partial derivatives can be computed as

$$D_{xx} = \frac{\partial^2 f}{\partial x^2} \approx \frac{f(x_0+\epsilon)-2f(x_0)+f(x_0-\epsilon)}{\epsilon^2} \tag{9}$$

As previously mentioned, the implicit representation can be defined as a function $f(x,y,z)$ and thus can a quadric function be described as

$$\begin{aligned} f(x,y,z) = Ax^2 &+ 2Bxy + 2Cxz \\ &+2Dx + Ey^2 2Fyz \\ &+2Gy + Hz^2 + 2I^2 \\ &+J \end{aligned} \tag{10}$$

or on matrix form

$$\mathbf{p}^T\mathbf{Q}\mathbf{p} = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{11}$$

The normal for an implicit surface is defined in equa- tion 12 above. Since a quadric surface is known analytically, the differentiation can be applied to the quadric with coefficient matrix Q directly. This gives an analytic expression involving only the same coefficients.

$$\nabla f(x, y, z) = 2 \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 2\mathbf{Q}_{sub}\mathbf{p} \tag{12}$$

3 Results

In this section detailed information is provided regarding the output of the lab, where implicit surfaces were implemented. The implementations of the implicit functions were quite straightforward and easily implemented. The first task was to complement existing code to include all operators, operators such as union and intersection by implementing boolean operators on the iso-value of the two incoming implicit surfaces. If the minimum of the two was returned, the union was obtained (see Figure 3). If the maximum of the two was returned, intersection was obtained. A difference operator was implemented, which returned the maximum of implicit surfaces A_1 and $-A_2$. A sphere is visualized in Figure 4 and its scalar cut plane.

The second task was to implement the quadric implicit surfaces (Equations 11 and 12). Various different basic primitives, such as sphere, cone or ellipsoid, with the quadric implicit functions were also defined in *FrameMain*. A hhyperboloid can be studied in Figure 4 and 6 with its gradients and mesh normals.

4 Lab partner and grade

My lab partner was Viktor Tholén, and I am aiming for grade 3.

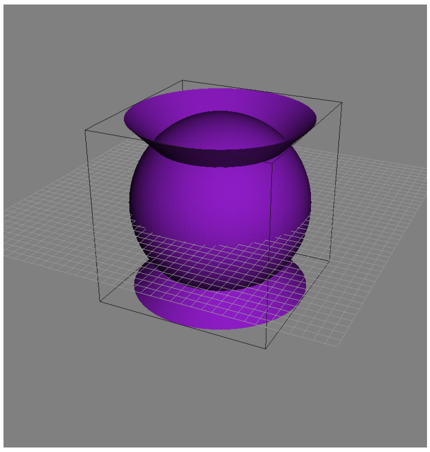

Figure 1: Two implicit surfaces, a cone and a sphere

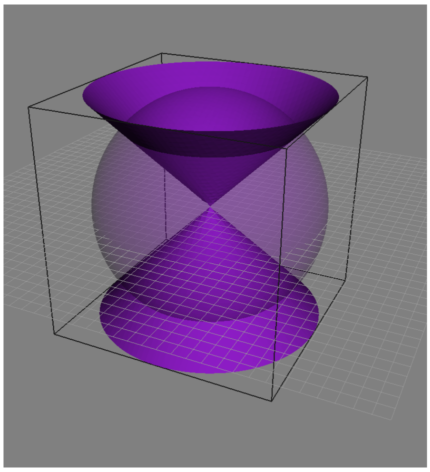

Figure 2: Two implicit surfaces, a cone and a sphere.

References

[1] Emma Broman Mark Eric Dieckmann, Robin Skånberg. *Implicit surfaces*. Modeling and animation, lab 4, 2021.

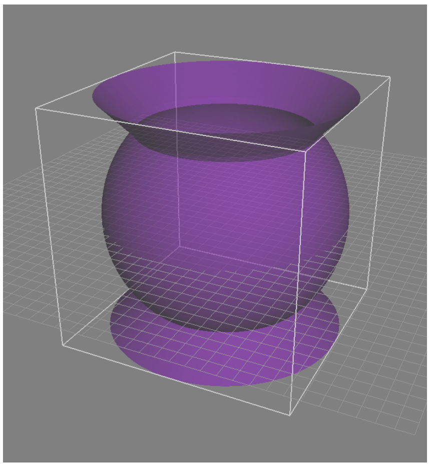

Figure 3: Two implicit surfaces, a cone and a sphere, and the union of the two.

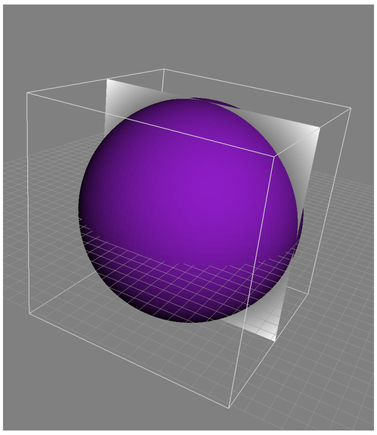

Figure 4: An implicit sphere and its scalar cut plane.

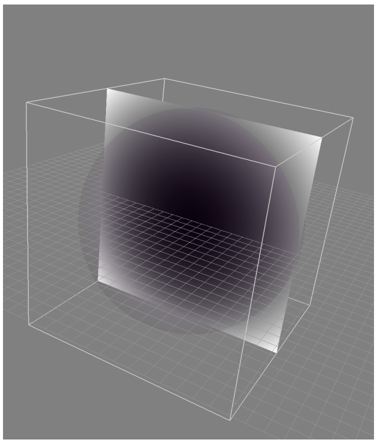

Figure 5: An implicit sphere and its scalar cut plane.

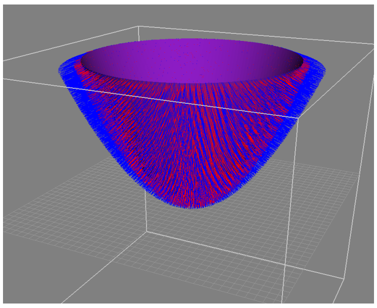

Figure 6: A hyperboloid and its gradients and mesh normals.