LAB REPORT: LAB 3

TNM079, MODELING AND ANIMATION

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Abstract

This is the third lab in a series of six labs in the course TNM079 Modeling and Animation at Linköping University. The report covers the topic of smooth surfaces and curves in computer graphics. Subdivision curves and surfaces are studied as an approximation thereof.

1 Introduction

In computer graphics, many applications use subdivision surfaces and meshes because they are computed relatively fast. The operations on the splines only affect the curve locally, and the computational cost is low because the order of the splines can be kept relatively low. Since all calculations are based on stable bases the numerical scheme is stable. Subdivision curves are based on spline theory, and this report only focuses on the assumption of uniform basis functions. This allows for simpler analysis as well as construction, but does limit some geometric properties.

2 Background

In this lab and report, work is carried out with parametric representations. A Bézier curve is a parametric curve used in computer graphics and related fields. The curves are named after Pierre Bézier, who used it in the 1960s for designing curves for the Renault cars. Other uses include the design of computer animation and fonts. Bézier curves can be used to form a Bézier spline, or generalized to higher dimensions to create Bézier surfaces. Given points p_0 and p_1, a linear Bézier curve is a straight line between p_0 and p_1. The curve is given by Equation 1.

$$\mathbf{p}_1(t) = (1-t)c_0 + tc_1, t \in [0,1] \tag{1}$$

where c_0 and c_1 are coefficients. The curve can be seen as a sum of coefficients and the basis function $(1-t), t$.

$$\begin{aligned} \mathbf{p}_2(t) &= (1-t)\mathbf{p}_{0,1}(t) + t\mathbf{p}_{1,1}(t), t \in [0,1] \\ &= (1-t)^2 c_0 + 2t(1-t)c_1 + t^2 c_2, t \in [0,1] \end{aligned} \tag{2}$$

A more common implementation is called quadratic Bézier curves, which uses three control points for the approximation of curve points. A quadratic Bézier curve us path traced by the function $\mathbf{p}(t)$, given c_0, c_1 and c_2. The function is a convex combination of two linear Bézier curves, where one, called $\mathbf{p}_{0,1}(t)$, is from c_0 to c_1, and the other, called $\mathbf{p}_{1,1}(t)$, is from c_1 to c_2 (see Equation 2).

As earlier mentioned the basis functions for linear interpolation are $(1-t), t$, where $t \in [0,1]$. In these functions the support is local, which means that only a finite number of basis functions are non-zero. Given a set of coefficients $\mathbf{c}$ and basis functions $b_i(t)$, the value of a function can be computed (see Equation 3).

$$\mathbf{p}(t) = \sum_i \mathbf{c}_i b_i(t) \tag{3}$$

In the case of the basis functions becoming zero, the resulting contribution of the sum disappears.

For the cubic case concerning basis functions a cubic basis function is considered on the interval $[-2, 2]$.

$$B_{i=0.3}(t) = \frac{1}{6}\begin{cases} (t+2)^3, \\ -2 \leq t < -1 \\ \\ -3(t+1)^3 + 3(t+1)^2 + 3(t+1) + 1, \\ -1 \leq t < 0 \\ \\ 3t^3 - 6t^2 + 4, \\ 0 \leq t < 1 \\ \\ -(t-1)^3 + 3(t-1)^2 - 3(t-1) + 1, \\ 1 \leq t \leq 2 \\ \\ 0, otherwise \end{cases} \tag{4}$$

The basis function in Equation 11 is called the cardinal cubic B-spline basis function. This B-spline fulfills the convex combination requirements and therefore forms a stable basis. A cubic uniform spline curve can be evaluated similarly to the Bézier curve as the sum of the products formed by the coefficients and the corresponding basis functions.

Construction by convolution is an interesting trait of the B-spline. By recursive convolution, a B-spline of degree d can be found. This can be done by convolving B_{d-1} with $B_0(4)$ (see Equation 5).

$$B_d(t) = \int B_{d-1}(s)B_0(t-s)ds \tag{5}$$

B-splines are refineable, which provides connection to subdivision methods. The equation is

$$B_d(t) = \frac{1}{2^d}\sum_{i=0}^{d+1}\binom{d+1}{i}B_d(2t-i) \tag{6}$$

where the binomial calculation is

$$\binom{k}{m} = \frac{k!}{m!(k-m)!} \tag{7}$$

By studying Equation 6 one can deduce that the B-spline of the degree 1 can be formulated as a linear combination of translated (i) as well as dilated $(2t)$ copies of itself. The continuity of the spline is increased by one for each refinement, and for a cubic B-spline the refinement will be

$$\begin{aligned} B_3(t) = \frac{1}{8}(1B_3(2t) \\ +4B_3(2t-1) \\ +6B_3(2t-2) \\ +4B_3(2t-3) \\ +1B_3(2t-4)) \end{aligned} \tag{8}$$

To express a geometric curve in terms of B-splines one can use the following equation:

$$\mathbf{p}(t) = \mathbf{B}(t)\mathbf{C} = \mathbf{B}(2t)\mathbf{SC} \tag{9}$$

The result of this refinement is that the support will be half as wide and spaced twice as dense. In Equation 10, $c_i^{'}$ denotes the re-weighted value for the i:th coefficient, $c_{i+\frac{1}{2}}^{'}$ denotes the new coefficient which corresponds to the refined basis function. The refined basis function is created between coefficients c_i and c_{i+1}.

$$\begin{aligned} c_i^{'} &= \frac{1}{8}(1c_{i-1} + 6c_i + 1c_{i+1}) \\ c_{i+\frac{1}{2}}^{'} &= \frac{1}{8}(4c_i + 4c_{i+1}) \end{aligned} \tag{10}$$

Loop subdivision scheme is a mesh refinement approach [1]. Each triangle in the original mesh is split into four new triangles. Then the new vertex positions are weighted averages of the old ones. β is calculated as

$$\beta = \begin{cases} \frac{3}{8k}, k > 3 \\ \frac{3}{16}, k = 3 \end{cases} \tag{11}$$

where k is the valence of the current vertex point. We also have the boundary constraints:

$$\begin{aligned} c_0^{'} &= c_0 \\ c_{end}^{'} &= c_{end} \end{aligned} \tag{12}$$

3 Results

In this section detailed information is provided regarding the output of the lab, where curve and mesh subdivision was implemented.

3.1 Curve subdivision

The task was to find the new coefficients c after a subdivision had been processed, as well as secure that the subdivision did not break any of the boundary constraints. The equations used were Equations 11 and 12. The container holding all the old coefficients was iterated over, and new weighted positions were calculated. New coefficients are also inserted between every other old coefficient. The boundary constraints were handled as in Equation 12. The start point as well as the end point were handled before and after the loop respectively.

Figures 1, 2 and 3 shows curve subdivision, where the analytical spline is red and green curve is subdivided. The figure shows how the green curve created through uniform cubic splines converges to the analytical red curve as the subdivision scheme is applied.

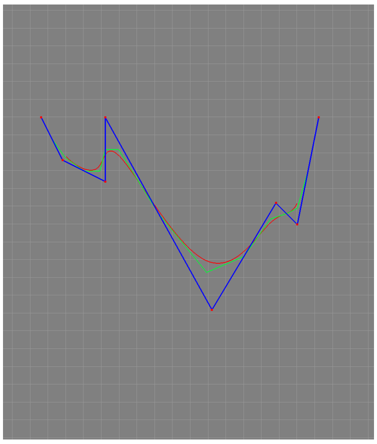

Figure 1: The analytical spline is red and green curve is subdivided once.

3.2 Mesh subdivision

The task was to implement weighting of the vertex rules of the new vertex points. For each vertex, a new position v was defined as the sum of all the weighted positions of the neighboring vertices. The positions were weighted with Equation 11. The

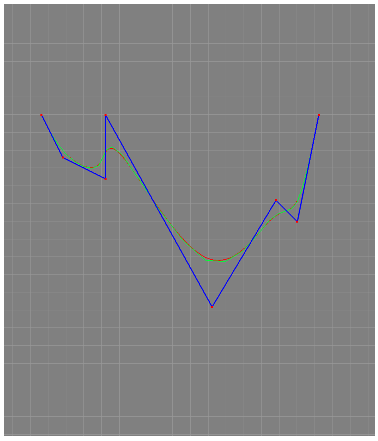

Figure 2: The analytical spline is red and green curve is subdivided twice.

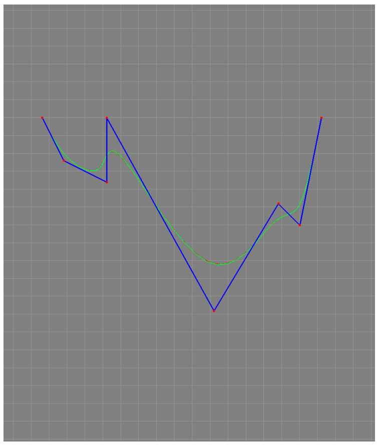

Figure 3: The analytical spline is red and green curve is subdivided thrice.

original position of the vertex was weighted with $(i - k\beta)$, where k is the valence (see Equation 13).

$$\overline{v'} = (1 - k\beta)\overline{v} + \sum_i \overline{v}_i\beta \tag{13}$$

As seen in Figures 4, 5 and 6, the details of the cow increase vastly for each subdivision. For each subdivision, the resolution is quadrupled.

4 Lab partner and grade

My lab partner was Viktor Tholén, and I am aiming for grade 3.

Figure 4: Mesh subdivision. Original mesh.

Figure 5: Mesh subdivision. One subdivision.

Figure 6: Mesh subdivision. Two subdivisions.

References

[1] C. Loop. Master's thesis. In *Smooth subdivision surfaces based on triangles*. Department of Mathematics, University of Utah, August 1987.

[2] Emma Broman Mark Eric Dieckmann, Robin Skånberg. *Splines and subdivision*. Modeling and animation, lab 3, 2021.